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Addendum

Canonical Products of Distributions and Causal Solutions of Nonlinear Wave Equations

This paper is a supplement to a previously published paper [1]. In [1] a class calculus for distributions had been introduced and applied to obtain causal solutions for a certain class of nonlinear wave equations. We shall complete this result here by proving a theorem on retarded and advanced causal solutions for these wave equations. For definitions and notations we refer to [1] (in [1, p. 405, line 14], read "$(m!/m_1! \cdots m_r!) \prod (D_j^{m_1}u_1, \ldots, D_j^{m_r}u_r)$" for "$\prod (D_j^{m_1}u_1, \ldots, D_j^{m_r}u_r)$.")

THEOREM. *Consider the nonlinear wave equation*

$$(\Box u + F(u)u)(x) = \delta(x), \qquad \Box = \partial^2/\partial t^2 - \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \partial^2/\partial x_3^2, \qquad x = (t, x_1, x_2, x_3), \tag{1}$$

where $F(u) = \sum_j F_j u_j$, $F_j \in \mathbf{R}$, $j \in \mathbf{N}$. *Let* $\lambda = t^2 - r^2$, $r^2 = x_1^2 + x_2^2 + x_3^2$ *and let* v *be a real- or complex-valued function of* λ *such that*:

(i) $Lv(\lambda) \equiv 4\lambda d^2v(\lambda)/d\lambda^2 + 8\, dv(\lambda)/d\lambda + F(v(\lambda))\, v(\lambda) = 0$;

(ii) *v is holomorphic in a neighborhood of* $\lambda = 0$;

(iii) $8\pi v_0 + F(v_0) = 0$, $v_0 \equiv v(0)$.

Let $\Theta_+(\lambda) \equiv \Theta(t - r)$ *and* $\Theta_-(\lambda) \equiv \Theta(t + r)$ *be the characteristic functions of* $\{(t, r) \mid t \geqslant r\}$ *and* $\{(t, r) \mid t \leqslant -r\}$, *respectively, and let* $\delta_+(\lambda) \equiv \delta(t - r)/2r$ *and* $\delta_-(\lambda) \equiv \delta(t + r)/2r$. *Then*

$$u_+(x) = \Theta_+(\lambda)\, v(\lambda) + (1/2\pi)\, \delta_+(\lambda) \tag{2a}$$

and

$$u_-(x) = \Theta_-(\lambda)\, v(\lambda) - (1/2\pi)\, \delta_-(\lambda) \tag{2b}$$

are maximal canonical retarded and advanced causal solutions of (1), *respectively, which are in the class* $\prod^*_{d,\mathbb{A},L}$.

Proof. If v satisfies (i) and (ii) then it follows with $\Box\Theta_+(\lambda) = 4\delta_+(\lambda)$ and $\Box\delta_+(\lambda) = 2\pi\delta(x)$ (cf. [2])

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$$
\begin{aligned}
\Box u_+(x) &= \Theta_+(\lambda)\,\Box v(\lambda) + 4v_0\delta_+(\lambda) + \delta(x) \\
&= \Theta_+(\lambda)[L - F(v(\lambda))]\,v(\lambda) + 4v_0\delta_+(\lambda) + \delta(x) \\
&= -\Theta_+(\lambda)F(v(\lambda))\,v(\lambda) + 4v_0\delta_+(\lambda) + \delta(x).
\end{aligned} \tag{3}
$$

As in [1, Example 14], one proves that

$$\textstyle\prod^*_{d,\mathbf{A},L}(F(u_+)u_+) = \{\theta_+F(v) + (1/2\pi)F(v_0)\,\delta_+\}. \tag{4}$$

Hence in order that (2a) be a maximal canonical solution of (1) we must have, due to (3) and (4),

$$4v_0\delta_+(\lambda) + (1/2\pi)F(v_0)\,\delta_+(\lambda) = 0,$$

which is equivalent to condition (iii). Using the relations (cf. [2]) $\Box\Theta_-(\lambda) = -4\delta_-(\lambda)$ and $\Box\delta_-(\lambda) = -2\pi\delta(x)$ the proof for (2b) is exactly as above. ∎

Example. The nonlinear wave equation

$$(\Box u + ku^3)(x) = \delta(x), \qquad k = \text{constant} > 0,$$

has exactly one retarded (advanced) maximal canonical solution in $\prod^*_{d,\mathbf{A},L}$. The retarded solution is given by

$$u_+(x) = -\pi\Theta_+(x)(\pi^2\lambda + k/8)^{-1} + (1/2\pi)\,\delta_+(\lambda),$$

the advanced solution by

$$u_-(x) = -\pi\Theta_-(x)(\pi^2\lambda + k/8)^{-1} - (1/2\pi)\,\delta_-(\lambda).$$

Proof. The only real solution of $v + kv^3 = 0$ which is holomorphic in a neighborhood of $\lambda = 0$ and satisfies

$$[8\pi v(x) + kv(x)^2]_{\lambda=0} = 0$$

is given by

$$v(\lambda) = -\pi(\pi^2\lambda + k/8)^{-1}. \quad ∎$$

Remark. The retarded and advanced solutions (3) and (6), respectively, could also have been obtained by the approximation procedure described in [3].

References

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